Mechanics of Wealth Dynamics with Heterogeneous Beliefs

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Abstract

This paper studies wealth dynamics in the environment with heterogeneous beliefs. We show that wealth transitions can be decomposed into two components. The first is a standard income hedging component. The second is a speculative trade and exists only due to differences in beliefs. We obtain conditions when belief heterogeneity amplifies or dampens wealth dynamics.

Asset prices are a function of the "average belief" – a wealth weighted CES aggregator of all agents' beliefs. And the speculative trade is proportional to the difference between an individual's and the average belief.

1 Introduction

2 Model

Time is discrete and indexed by $t \in 0, 1, 2, ...$ A non-storable good is traded in each period. The aggregate state of the economy is fully summarized by a stochastic process s_t. It is a stationary first-order Markov process with a finite set of states $\mathcal{S} = \{1, ..., S\}$ and a transition matrix $\Pi = \{\pi_{ij}\}$ where $\pi_{ij} = prob(z_{t+1} = j | z_t = i)$. We let s^t to denote the history of the state

$(s_0, s_1, ..., s_t)$. Agents' beliefs about the state of the economy differ. We denote the transition probability matrix of agent i by Π^i.

The economy is populated by a finite number $I \geqslant 2$ of infinitely lived agents. Agent i receives income $y^i(s^t)$ in period t and state s_t. The aggregate income is $y(s^t) = y_0 g(s_1) \cdots g(s_t)$ where $g(s_t)$ is a stochastic growth rate in state s_t. Individual income shares are functions of the current state. That is $y^i(s^t) = \eta^i(s_t) y(s^t)$. We assume that $g(s_t)$ and $\eta^i(s_t)$ are independent stochastic processes.

Financial markets trade a full set of Arrow securities.

Every agent manages his income to achieve the highest utility from consumption:

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^i(s^t) u(c_t^i(s^t)), \quad \beta \in (0, 1), \tag{1}$$

subject to a sequence of budget constraints:

$$c_t^i(s^t) + \sum_{s_{t+1}} Q_t(s^{t+1}) a_{t+1}^i(s^{t+1}) = y^i(s_t) + a_t^i(s^t), \quad \forall t \tag{2}$$

and *natural* borrowing limits:

$$a_{t+1}^i(s^{t+1}) \geqslant -\bar{B}^i(s^{t+1}), \quad \forall s_{t+1}. \tag{3}$$

These history dependent borrowing limits are the largest borrowing limits that never bind in equilibrium.

3 Two-period two-agent economy

We analyze a two-period two-agent model to build intuition about how a dynamic model works. We let the two individuals to be completely symmetric in period 0. Both agents receive one unit of good and start with zero assets:

$$(e^1(s_0), e^2(s_0)) = (1, 1), \quad (a^1(s_0), a^2(s_0)) = (0, 0).$$

Individual endowments are "symmetric"; so, instead of a competitive equilibrium problem we can solve a Pareto problem with equal weights on the two agents. We assume that $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$. Let $\bar{e}(s) = 0.5(e^1(s) + e^2(s))$ denote the per capita endowment in state s.

First, let's analyze the optimal allocation. Agent a consumes 1 unit of good in period 1 and

$$c_1^a(s) = \bar{e}_1(s)\frac{(\pi^a(s))^{1/\gamma}}{0.5(\pi^1(s))^{1/\gamma} + 0.5(\pi^2(s))^{1/\gamma}}$$

in period 1, state s. Agent 1's purchase of security s is

$$a^1(s) = (\bar{e}_1(s) - e_1^1(s)) + \bar{e}_1(s)\frac{(\pi^1(s))^{1/\gamma} - (\pi^2(s))^{1/\gamma}}{(\pi^1(s))^{1/\gamma} + (\pi^2(s))^{1/\gamma}}$$

The first term is standard and reflects desire to hedge income risk. The second term is speculative trade,

$$\textit{speculative trade} := \bar{e}_1(s)\frac{(\pi^1(s))^{1/\gamma} - (\pi^2(s))^{1/\gamma}}{(\pi^1(s))^{1/\gamma} + (\pi^2(s))^{1/\gamma}},$$

which is proportional to the difference in beliefs adjusted for risk attitude. When do the hedge and the speculative trade have the same sign? When an agent is pessimistic. That is he must put a lower probability than the other agent on states when his or her income is above the population average.

Result 1. In a two-period economy:
a) Asset trades can be decomposed into a hedging and a speculative component;
b) Speculation amplifies/dampens asset trades when agents are pessimistic/optimistic.

Second, let's analyze the equilibrium price system. Price of an Arrow security paying is state s is

$$Q(s) = \beta(\bar{e}_1(s))^{-\gamma}\tilde{\pi}(s), \tag{4a}$$

where

$$\tilde{\pi}(s) = [0.5(\pi^1(s))^{1/\gamma} + 0.5(\pi^2(s))^{1/\gamma}]^{\gamma}, \tag{4b}$$

a CES aggregator of agents' beliefs. The return of a risk-free bond and expected return on a claim to the aggregate endowment are

$$R^f = \beta^{-1}\frac{1}{\sum_s(\bar{e}_1(s))^{-\gamma}\tilde{\pi}(s)}, \tag{5a}$$

$$E[R^e] = \beta^{-1}\frac{\sum_s \pi(s)\bar{e}_1(s)}{\sum_s(\bar{e}_1(s))^{1-\gamma}\tilde{\pi}(s)}. \tag{5b}$$

What system of beliefs is needed to solve the asset pricing puzzles? We need agents to be pessimistic about aggregate states of the world and $\gamma < 1$. Note that heterogeneity of beliefs across idiosyncratic states does not matter in this two period example. It matters in a multiperiod setting because it affects the weights of agents' beliefs in $\tilde{\pi}$.

Result 2. In a two-period economy:
a) Risk-free rate is low/high when individuals are pessimistic/optimistic;
b) Equity return is high when individuals are pessimistic and risk-aversion is low or individuals are optimistic and risk-aversion is high.

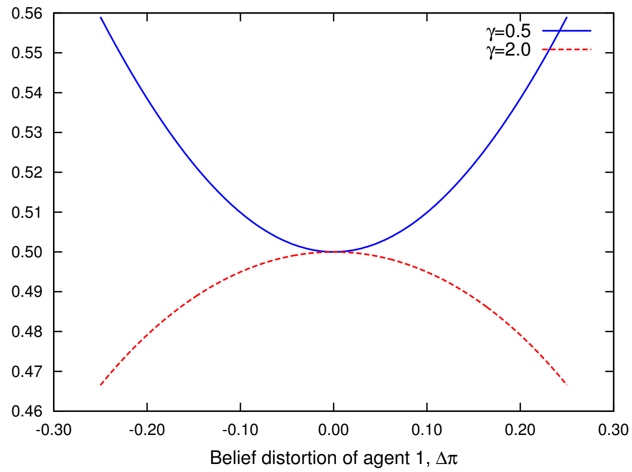
Returning to the beliefs aggregator, note that $\gamma \geqslant 0$.[1] So, depending on γ belief aggregator $\tilde{\pi}$ can be either concave or convex. When $\gamma < 1$ the aggregator is concave in individual beliefs. That is more dispersed individual beliefs will lead to a pessimistic aggregator. The opposite is true when $\gamma > 1$. With logarithmic preferences an increase in belief dispersion leaves asset prices unchanged. Lastly, when beliefs are correct on average, $(\pi^1(s), \pi^2(s)) = (\pi(s) + e, \pi(s) - e)$ where e lies in a small open interval centered at 0, the effect on asset prices is $O(|e|^3)$. See also figure 1. In such cases asset price dynamics can be attributed entirely to wealth distribution dynamics.

4 Computation of competitive equilibrium

We exploit the fact that a competitive equilibrium solves a Pareto problem. We use a modified Negishi algorithm that replaces Pareto weights with the initial distribution of consumption. We derive the relation that allows us to compute recursively consumption plan of each agent for any initial distribution of consumption. The consumption plans, in turn, can be used to compute the price system. Then the consumption plans and the price system can be used to back out asset-trading plans. The last step provides us with the initial wealth distribution that supports the computed allocation as a CE. This relation can be inverted to compute a CE for any initial wealth

[1] CES aggregator considers $\gamma \in (-\infty, 1]$.

Figure 1: Belief aggregator for "symmetric" beliefs



distribution.[2]

We could compute the competitive equilibrium allocations as a function of the Pareto weights.[3] Our method allows computing agents' trading strategies. We are also able to compute the initial distribution of wealth that supports a given competitive equilibrium.

4.1 Consumption dynamics and asset prices

First, we derive the equation that describes dynamics of individual consumption. For this purpose we use the fact that the natural borrowing limits that the agents face never bind. We solve the following system of equations:

$$\begin{aligned} Q(s^{t+1}|s^t) &= \beta g(s_{t+1})^{-\gamma}\pi^i(s'|s)[\hat{c}^i(s^{t+1})/\hat{c}^i(s^t)]^{-\gamma}, \quad \forall s \in S, i \in I, \\ 1 &= \sum_{i\in I} \hat{c}^i(s^t), \\ 1 &= \sum_{i\in I} \hat{c}^i(s^{t+1}). \end{aligned}$$

[2]With symmetric agents the computation is trivial as we can use the fact that each agent is equally weighted in the Pareto problem.

[3]See for example Cogley-Sargent 2009, AEJ.

The above system has the following solution:

$$\hat{c}^i(s^{t+1}) = \frac{\hat{c}^i(s^t)[\pi^i(s_{t+1}|s_t)]^{1/\gamma}}{\sum_{j\in I}\hat{c}^j(s^t)[\pi^j(s_{t+1}|s_t)]^{1/\gamma}}, \quad \forall i \in I, \tag{6a}$$

$$Q(s^{t+1}|s^t) = \beta g(s_{t+1})^{-\gamma}\left[\sum_{j\in I}\hat{c}^j(s^t)[\pi^j(s_{t+1}|s_t)]^{1/\gamma}\right]^{\gamma}. \tag{6b}$$

Equation (6a) describes dynamics of agent i's consumption share. Consumption share of agent i increases (decreases) when she assigns higher (lower) probability to event s_{t+1} than the CES aggregate of the population beliefs.

Result 1. Agent i's consumption increases during the transition $s^t \to s_{t+1}$ when the probability that she assigns to the transition is larger than the CES aggregate of the population beliefs:

$$\hat{c}^i(s^{t+1}) > \hat{c}^i(s^t) \leftrightarrow \pi^i(s_{t+1}|s_t) > \tilde{\pi}(s_{t+1}|s_t) \equiv \left[\sum_{j\in I}\hat{c}^j(s^t)[\pi^j(s_{t+1}|s_t)]^{1/\gamma}\right]^{\gamma}.$$

Interestingly, whether consumption of agent i is going to increase or decrease depends only on the subjective beliefs and current consumption distribution. In particular, it does not depend on the distribution of income. The latter only impacts the asset positions that agents take.

If the agents had the same beliefs (not necessarily correct) consumption share would remain constant. This is a standard result for time-separable homothetic welfare function.

Equation (6b) shows that Arrow security (state) prices take the standard form for time separable CRRA welfare function. It is a product of the discount factor β, aggregate marginal utility growth $g(s_{t+1})^{-\gamma}$ and the conditional "probability" of the state $\tilde{\pi}(s_{t+1}|s_t)$. Unless $\gamma = 1$ the CES belief aggregate $\tilde{\pi}(s_{t+1}|s_t)$ is not a probability distribution in general. If the agents held the same (not necessarily correct) beliefs the CES term would reduce to the common probability $\pi(s_{t+1}|s_t)$.

Result 2. Arrow security (state) prices are functions of the CES aggregate of the population beleifs:

$$Q(s^{t+1}|s^t) = \beta g(s_{t+1})^{-\gamma}\tilde{\pi}(s_{t+1}|s_t).$$

As is evident from equations (6a) and (6b) the sufficient state for this economy is the current exogenous state s_t and the distribution of consumption shares. The latter is a function of financial wealth in the economy. Computation of financial wealth distribution that supports a given distribution of consumption is the subject of the next section.

Result 3. Competitive equilibrium allocation and the price system are functions of the consumption, and hence financial wealth, distribution.

4.2 Values of consumption and income streams

We now derive the map between the distribution of consumption and wealth. Let $P_c^i(\hat{c}, s)$ be the *normalized* (per unit of aggregate endowment) value of consumption stream of agent 1 conditional on her consumption share (in the aggregate endowment) being $\hat{c}$ and state of the economy s. P_c^i can be represented recursively as follows:

$$P_c^i(\hat{c}, s) = \hat{c} + \sum_{s'} Q(s'|\hat{c}, s) P_c^i(\eta^i(s'|\hat{c}, s), s') g(s'), \tag{7}$$

where $\hat{c} = (\hat{c}^1, ..., \hat{c}^I)$ and

$$\eta^i(s'|\hat{c}, s) = \frac{\hat{c}^i(\pi^i(s'|s))^{1/\gamma}}{\sum_{j\in I} \hat{c}^j(\pi^j(s'|s))^{1/\gamma}},$$

$$Q(s'|\hat{c}, s) = \beta(g(s'))^{-\gamma} \left[\sum_{j\in I} \hat{c}^j(\pi^1(s'|s))^{1/\gamma}\right]^{\gamma}.$$

are stationary versions of (6a) and (6b). Similarly, normalized value of agent i's income $P_e^i(c, s)$ must satisfy the following recursive equation:

$$P_e^i(\hat{c}, g) = \hat{y}^i(s) + \sum_{s'} Q(s'|\hat{c}, s) P_e^i(\eta^i(s'|\hat{c}, s), s') g(s'). \tag{8}$$

The date-0 budget constraint then (implicitly) defines a map between current assets and current consumption $\hat{c} = \rho(\hat{a}, s)$:

$$\hat{a}^i = V_c^i(\hat{c}, s) - V_e^1(\hat{c}, s), \quad \forall i \in I. \tag{9}$$

If we want to compute the competitive equilibrium for the case with $(a_0^1, ..., a_0^I) = (0, ..., 0)$ we need to start from the initial consumption distribution $\hat{c}_0$ that solves $V_c^i(\hat{c}_0, s_0) = V_e^1(\hat{c}_0, s_0), \forall i \in I$.

4.3 Learning from prices?

Can individuals learn that their beliefs are different from the rest by observing equilibrium prices? The answer is suggested by equation (6b). One needs $|S|(|S|-1)(I-1)$ restrictions[4] to identify beliefs of the others. With $|S|$ bits of information ($|S|$ Arrow security prices) arriving each period complete learning is possible after $(|S|-1)(I-1)$ periods. With an infinite number of states or an infinite number of agents, as in e.g. overlapping generations model, learning would be impossible.

We illustrate information revelation in the economy with three agents and two exogenous states. We also assume that preferences are logarithmic. Then agent i gets the following information from prices:

$$Q(1|\hat{c}, s)/[\beta g_1^{-\gamma}] = \hat{c}^1\pi^1(1|s) + \hat{c}^2\pi^2(1|s) + (1 - \hat{c}^2 - \hat{c}^3)\pi^3(1|s),$$
$$Q(2|\hat{c}, s)/[\beta g_2^{-\gamma}] = \hat{c}^1\pi^1(2|s) + \hat{c}^2\pi^2(2|s) + (1 - \hat{c}^2 - \hat{c}^3)\pi^3(2|s).$$

This is a system of two equations in two unknowns $\pi^2(1|s), \pi^3(1|s)$ that has a unique solutions. So, as soon as both states realize agents will be able to determine each other's transition matrices. This can happen as soon as after two periods.

Learning from outcomes on the other hand eventually leads to homogeneous (but possibly incorrect) beliefs. Such environments are studied in Cogley-Sargent and Cogley-Sargent-T.

5 Simulation results

5.1 Parameterization

We are going to parameterize the model in the following way. We assume that an individual's income volatility is as observed in the U.S. during 1984-2007. Discount factor is chosen so that the expected risk-free rate is 1%.

[4] We have to solve for $I-1$ transition matrices, each with $|S|(|S|-1)$ parameters.

The mapping between the economy state and the aggregate endowment share and income distribution is:

$$(g(s), \eta(s)) = \begin{cases} (g_l, f_l, f_h), & s = 1 \\ (g_l, f_h, f_l), & s = 2 \\ (g_h, f_l, f_h), & s = 3 \\ (g_h, f_h, f_l), & s = 4 \end{cases}.$$

We choose g_l, g_h, f_l, f_h together with the transition matrix Π to match the following facts:
F1. Mean growth rate is 1.030.
F2. Volatility of growth rate is 0.030.
F3. Expansions are 2.65 times more likely.
F4. Persistence and volatility of individual income shares are $(0.530, 0.296)$.

5.2 Beliefs

We allow agent to disagree about the income distribution process but not about the aggregate growth rate. The transition matrix of agent i is:

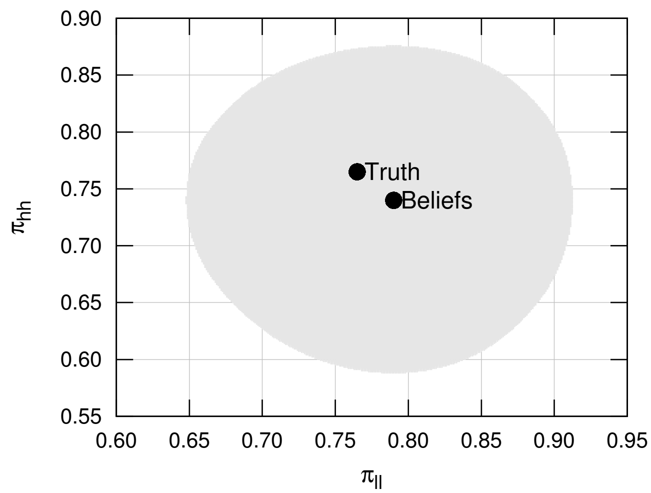
$$\Pi^i = \begin{bmatrix} \pi_{ll}^g & \pi_{lh}^g \\ \pi_{hl}^g & \pi_{hh}^g \end{bmatrix} \otimes \begin{bmatrix} \pi_{ll}^\eta - \Delta p^i & \pi_{lh}^\eta + \Delta p^i \\ \pi_{hl}^\eta - \Delta p^i & \pi_{hh}^\eta + \Delta p^i \end{bmatrix},$$

where $\Delta p^1 = -\Delta p^2$ measures the degree of i's pessimism. If $\Delta p^1 = -\Delta p^2 < 0$ then agent 1 assigns higher than the true probability to states in which his income share is low. Since the income share and the aggregate growth rate processes are independent we can easily measure agent's ability to identify their errors from data of a given length. Assume that the agent has a dataset with $T = 100$ observations of η_t. We ask what is the probability that she is going to reject her beliefs using a likelihood ratio test with confidence $\alpha = 0.50$. The answer is easiest to illustrate graphically. Figure 2 plots the region of $(\hat{\pi}_{ll}, \hat{\pi}_{hh})$ that would be doubted with a sample of 100 observations α fraction of samples.

5.3 Numerical results

Next we present simulated series of consumption shares and financial wealth. The dashed line is the solution for the economy with homogeneous beliefs.

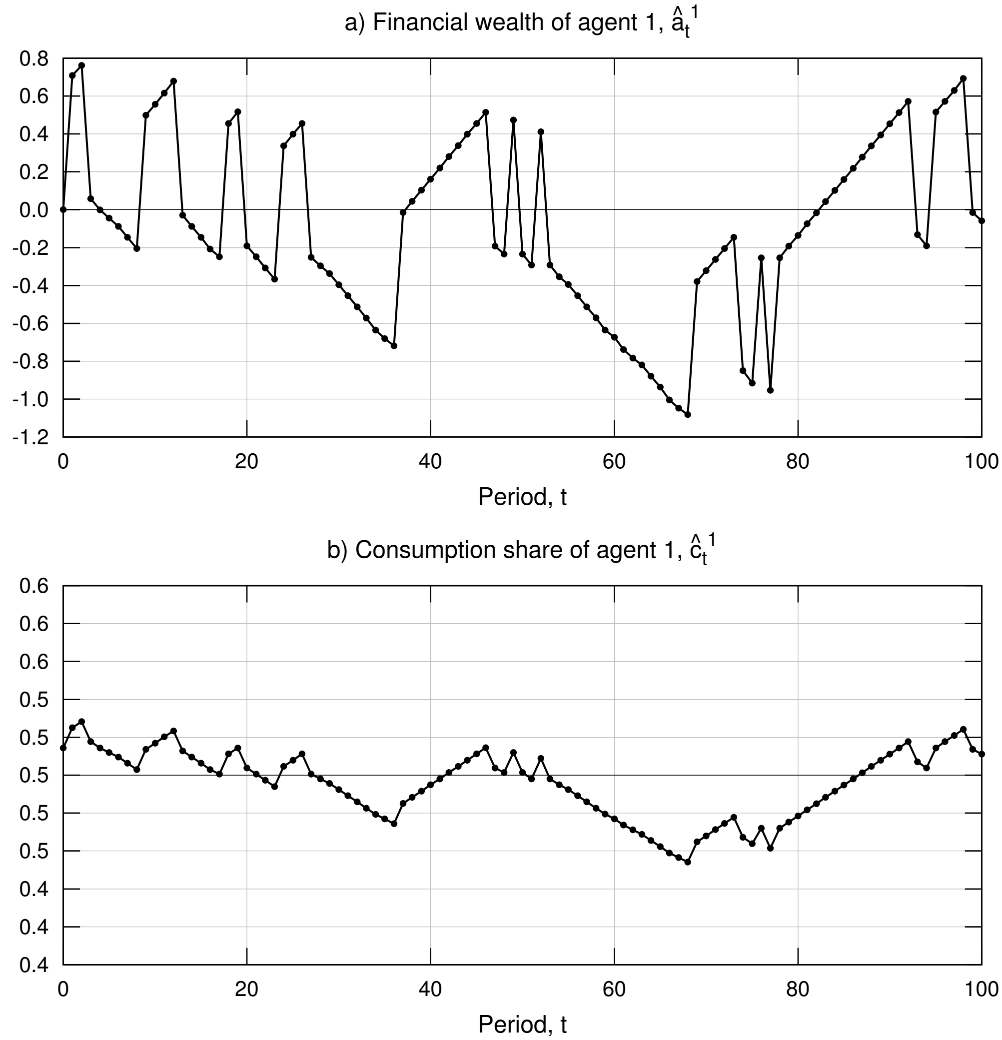
Figure 2: Belief error detection, $\Delta p^1 = -0.025$



Conveniently, in our parametrization the true belief is also the (unweighed) average population belief. In this case financial wealth distribution can take only 4 values. As Judd,... if financial markets traded long-term assets portfolios would be constant. With heterogeneous beliefs things change dramatically. Financial wealth now varies significantly. Agent 1's financial wealth $\hat{a}^1$ increases beyond what is predicted by the homogeneous beliefs case when agent 1 is pessimistic. The effect of heterogeneous beliefs is larger when economy expands as more resources can be committed in a transaction. Turning to consumption, it is predicted to be constant in the homogeneous beliefs case. But with heterogeneous beliefs consumption of an agent increases in states that she assigns higher probability than others. This increase in consumption is supported by returns from speculative trading.

We now turn to the wealth distribution. How quickly does it fan out? We plot in figure 4 the empirical distribution of $\hat{a}^1_{100}$. It is computed from $N = 100,000$ simulations using $\sqrt[3]{N}$ bins. The initial state in each simulation is $(\hat{a}^1_0, s_0) = (0, 1)$. Two observations emerge from the figure. First, it does not matter much wether agents are pessimistic or optimistic. The reason for this is that speculative trades are substantially larger than trades hedging income. And the size of speculative trades depends only on relative

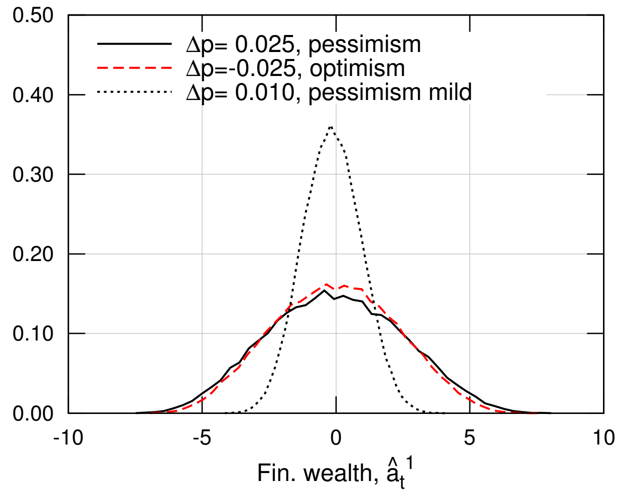
Figure 3: Simulated path of wealth and consumption



belief differences and not on the sign of the errors. When $\Delta p = 0$ wealth distribution degenerates to four – one for each state – points.

Figure 5 plot the realized return on the equity share that pays the aggregate endowment and the return on the risk-free rate. The most interesting feature of the simulation is high volatility of equity returns in periods 40-50 and 75-85. This happens when wealth is approximately equally distributed in

Figure 4: Financial wealth for different specifications of beliefs



the economy and agents luck switches. States when agent 1 (or 2) looses high pay are also states when she gets speculative returns. Substantial amount of wealth reallocation creates extra volatility in the equity markets. Volatility in the equity markets is large if no agent has large debt initially as in this case speculative trades are larger.

6 Long run risk

Slowly moving wealth distribution = long run risk?

Figure 5: Return on equity and risk-free debt