Asset Markets, Diverse Beliefs, and Borrowing Limits

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Abstract

This paper studies asset prices, trading volume and wealth dynamics in the environment with heterogeneous beliefs and incomplete markets. Agents can trade the full set of Arrow securities but are subject to, endogenous or exogenous, borrowing limits. Borrowing limits are always active because differences in opinions lead to a highly volatile wealth. Further, no agents are driven out of the financial markets and this insures existence of a well-defined ergodic joint distribution of asset returns, trading volume and macroeconomic fundamentals.

We assume small divergence of opinions about idiosyncratic income processes, rather than disagreement about aggregate processes as assumed in other studies. This improves the model's asset pricing predictions and generates volatility clustering. Trading volume is persistent, volatile and uncorrelated with asset returns or the fundamentals.

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1 Introduction

Since Lucas (1978) and Mehra & Prescott (1985) many papers were written suggesting a solution to the asset pricing puzzles. Most of these models are single agent models. This means that there is no trading in equilibrium and, so, asset prices are an 'off equilibrium' phenomenon. The only role that they play is to create expectations that trading in financial markets is unprofitable. But asset prices and trading volume are determined jointly in equilibrium, the point stressed by Blume et al. (1994). Economists have long tried to match prices (stock returns) and quantities (consumption). But the true quantity – trading volume in financial markets – has been largely ignored.

Trading volume has been studied before in the context of individual asset markets, *e.g.* He & Wang (1995). These studies relied on partial equilibrium models that are better suited to address such phenomena as, say, momentum. Yet, wealth dynamics in those models is largely ignored. That is agents could accumulate arbitrarily large negative positions. In general equilibrium agents would instead be driven out of the financial markets as conjectured by Friedman (1953) and shown by Sandroni (2000) and Blume & Easley (2006).

In this paper we build a general equilibrium model of an exchange economy with diverse beliefs as in Harrison & Kreps (1978). Asset prices in our model depend on the wealth distribution among agents. Agents that have more accurate beliefs or are simply lucky accumulate wealth and prices increasingly reflect their beliefs. Individuals that have less accurate beliefs or are unlucky de-accumulate wealth. However, agents are subject to borrowing limits and this precludes them from loosing all their wealth. Thus, agents are able to recover from financial loses and, while some markets may occasionally close, aggregate trading activity never ceases.

In the model agents trade for two reasons: hedging and speculation. For a typical macroeconomic parametrization trading for hedging purposes is very limited. Judd et al. (2003) show that trading volume in the model with complete financial markets is small. Speculative trading, on the other hand, can be significant even with small differences in opinions. Speculation triggers substantial wealth transfers. Asset prices tend towards valuation of the agent gaining wealth. Such periods are associated with high excess returns and increased trading activity. Trading activity is subdued when wealth is distributed unevenly, that is some agents are close to their borrowing limits.

Yet, equity returns can be either high or low depending on the configuration of beliefs. When agents have correct beliefs about the aggregate growth process asset prices and returns are low when wealth is distributed unevenly. Endogenous movement of wealth distribution generate volatility clustering in asset prices and trading volume as observed in the data. It also accounts for a disconnect between asset market dynamics and the macroeconomic fundamentals.

In this work agents disagree about their individual income processes. Unlike in other studies, everyone agrees on the evolution of the aggregate growth rate. We assume that agents do not update their beliefs. But we specify individual beliefs so that an econometrician would not be able to reject them using 100 years worth of data. It is possible to incorporate learning from individual past experience. But even in simple setups beliefs can stay away from truth for protracted periods of time. For example, Cogley et al. (forthcoming) show that dynamics with dogmatic beliefs and with learning can be very similar.

Our model also allows for disagreement about the aggregate processes. Provided that it is not the only source of disagreement, divergence of opinions about the disaster risk also improves the model's asset pricing predictions along many dimensions.

Finally, we examine endogenous borrowing limits as in Kehoe & Levine (1993), also called "solvency constraints" by Alvarez & Jermann (2001). Yet, we find that without reducing the exclusion period following default such borrowing limits are unreasonably tight. Despite this fact belief diversity allows the model's stochastic discount factor to pass Hansen-Jaganathan's bound test for a much larger set of parameters.

2 Aggregate trading activity

De facto standard for measuring market activity is stock turnover. Let n_t^i denote turnover of stock i in period t. Let q_t^i and q_t be respectively the price of stock i and the value of all traded stocks (stock market capitalization) in period t. Value weighted stock market turnover is:

$$\textit{Aggregate turnover}_t = \sum_{i \in A_t} \frac{q_t^i}{q_t} n_t^i,$$

where A_t denotes the set of all stocks traded (active) in period t. Note that this measure is simply the ratio of the value of all market transactions in period t to the total market capitalization. Because later we turn to growth rates we use the value of all transactions in the market:

$$\textit{Aggregate value of trades}_t = \sum_{i \in A_t} q_t^i n_t^i. \tag{1}$$

Figure 1 plots the value of all transactions in any given month and the total market capitalization. All recorded transactions from the American stock exchanges are included. The American stock market accelerated in early 1990s. In 1997 the value of all transactions outpaced market capitalization. That is trading activity grew beyond what was implied by the growth of the market value of the traded companies.

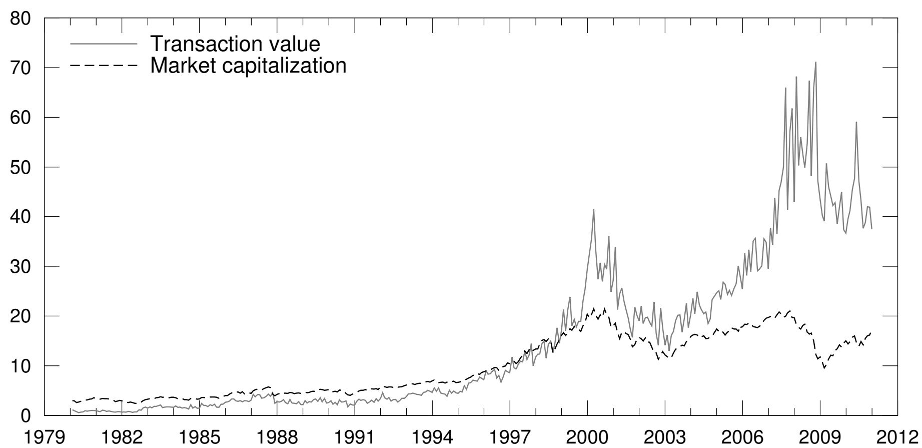


Figure 1: Trading activity and market size, trln real 2005$

Figure 2 plots the growth rates of the transaction value, the market capitalization and the industrial production. Standard deviation of the first two series is 17.32% and 4.64% respectively. The correlation between the two series is 0.2879. When we restrict our attention to 1999-2010, a period of high stock market turmoil, this correlation drops to 0.1466 and is statistically insignificant. Correlation between the financial series and industrial production, -0.8% and -5.9% respectively, is statistically indistinguishable from zero.

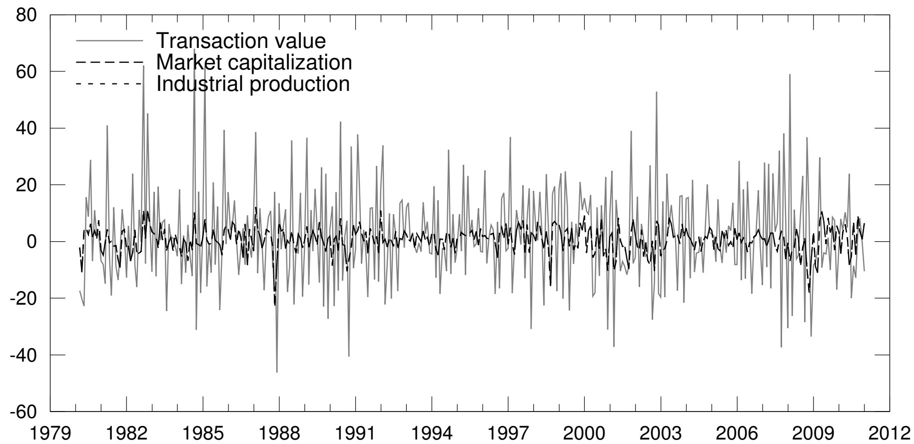


Figure 2: Trading activity and market capitalization growth rates, %

3 Model

Time is discrete and indexed by $t \in \{0, 1, 2, ...\}$. There is one perishable good in each date. The state of the economy is a first-order Markov process s_t with a finite set of states $\mathcal{S} = \{1, ..., S\}$ and a transition matrix $\Pi = \{\pi_{ij}\}$ where $\pi_{ij} = prob(s_{t+1} = j | s_t = i)$. We let s^t to denote the partial history of the state $(s_0, s_1, ..., s_t)$. Aggregate endowment grows at a stochastic rate: $y(s^t) = y_0 g(s_1) \cdots g(s_t)$.

The economy is populated by a finite number I of infinitely lived agents. Agent i receives income $y^i(s^t)$ in period t after history s^t. Individual income shares are functions of the current state only. That is $y^i(s^t) = \eta^i(s_t) y(s^t)$. We assume that $g(s_t)$ and $\eta^i(s_t)$ are independent stochastic processes.

Agent i can trade a full set of Arrow securities at each date and history subject to a set of borrowing limits $B^i(s^t)$. Initial distribution of assets $\{a_0^i(s_0)\}_{i=1}^I$ is fixed.

Agent i believes that the transition probability matrix of the economy's state is Π^i and chooses a consumption stream $\{c^i(s^t) : \forall t, s^t\}$ to achieve the highest subjective expected utility:

$$U(c^i | s_0) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c^i(s^t)) \pi^i(s^t | s_0), \quad \beta \in (0, 1), \tag{2}$$

subject to a sequence of budget constraints:

$$c^i(s^t) + \sum_{s_{t+1}} Q(s^{t+1}) a^i(s^{t+1}) = y^i(s_t) + a^i(s^t), \quad \forall t, s^t, \tag{3}$$

and borrowing limits:

$$a^i(s^{t+1}) \geqslant -B^i(s^{t+1}), \quad \forall t, s^{t+1}. \tag{4}$$

3.1 Endogenous borrowing limits

The model is well-defined for any set of borrowing limits that are at least as tight as the natural borrowing limits. This includes, as special case, the endogenous solvency constraints of Kehoe & Levine (1993) and Alvarez & Jermann (2001). These limits are defined implicitly: it is the borrowing limit at which an agent is different between repaying the debt and being banned from financial markets. An agent living in financial autarky has no choices to make and simply consumes his income every period. We denote the life-time utility of agent i who has just been banned from the financial market by $V^i_{aut}(s_t)$. It is a solution to the following recursive equation:

$$V^i_{aut}(s^t) = u(y^i(s_t)) + \beta \sum_{s_{t+1}} \pi^i(s_{t+1}|s_t) V^i_{aut}(s^{t+1}), \quad \forall t, s^t. \tag{5}$$

It is possible to allow agents returning to the financial market after a given number of periods. An agent can be also allowed into the market with some fixed probability. These possibilities enhance individuals' autarky value and, thus, further limit borrowing in equilibrium. We construct alternative reservation values when we state recursive formulation of the model.

3.2 Information and Learning

In our setting the assumption of common knowledge is unnecessary. We imagine an economy in which individual agents trade based on their beliefs and their expectations of future prices.

We assume that there is no learning: neither from state realizations nor from prices. This is justified on the grounds that our model is necessarily oversimplified. In a more realistic environment learning is a less natural assumption. But even if the agents were allowed to learn from past observations and prices their beliefs may not converge let alone converge to the true measure. As you will see later from formula (19) prices reveal only the average belief in the economy.[1] So, learning from prices must be limited. On the other hand, if one were allowed to see individual trades in addition to prices then it would be possible back out others' beliefs. This requires unreasonable amount of data collection and analysis on the part of individual agents. So, we assume that there is no learning from equilibrium prices.

On a technical side, we aim to obtain a stationary distribution of asset prices. With learning the effect of heterogenous beliefs would be only transient. For example, Cogley & Sargent (2009) study the transition path while agents are still learning the true distribution of the state. It is possible to force beliefs to fluctuate forever. Veronesi (2000) studies an economy with a hidden Markov state. An ever-changing unobserved regime prevents agents from learning the truth even in the limit. However, such an economy is a special case of a model with exogenously specified beliefs.[2] The reason is that the evolution of beliefs is still *exogenous* to the model.

4 Special Cases

This model includes as special cases two important classes of models. The first class of models features limited commitment. Agents are allowed to exit their market arrangements at a cost, usually temporary or perpetual loss of access to the market. This type of models is used to study incomplete risk-sharing. The second class of models has complete markets and heterogeneous beliefs. Agents take different asset positions because of their beliefs. This type of models is used to study survival of agents with incorrect beliefs.

4.1 Homogenous beliefs, endogenous borrowing limits

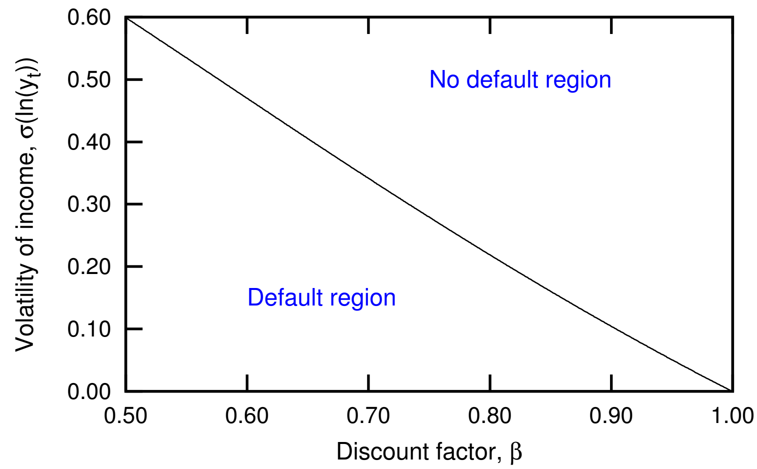
Kehoe & Levine (1993) and Kocherlakota (1996) study outcomes in the model with two sided lack of commitment.[3] Alvarez & Jermann (2000) show that this equilibrium can be decentralized by imposing on agents endogenous borrowing limits. The authors then calibrate the model to the U.S. data and

[1] If the economy is populated by two agents (not two types, each with continuum of agents) then it would be possible to back out private information of agents. Milgrom and Stokey arrive at the same conclusion.

[2] Though beliefs have to be modeled as a high order Markov process.

[3] Two-sided lack of commitment is fundamentally different from one-sided lack of commitment, *e.g.* Eaton-Gersovitz. In the latter an agent is allowed to renege on his promises to the principal/planner who is assumed to be risk-neutral.

Figure 3: Default region



demonstrate that the model can generate the high equity premium and the low risk-free rate close to those observed in the data.

In the equilibrium with limited commitment default option is never exercised. But the default constraint may or may not be binding. In the latter case, the economy's characteristics match those of the complete markets economy. In the case of homogeneous beliefs it is easy to characterize a set of parameters for which the constraint is binding. Figure 3 plots the default region for $u(c) = -1/c$ and an *iid* state s_t. Consistently with the results in Alvarez & Jermann (2000) the endogenous borrowing constraint is active when agents discount future heavily (β is low) or when agents face a significant amount of idiosyncratic risk (e is high). In both cases the value of the autarky increases relative to the arrangement with full risk-sharing. Note that for the degree of volatility in the individual data, $\sigma(log(y_t^i)) \approx 0.30$ to have the constraint bind in equilibrium one needs to assume an unconventionally low discount factor. The latter in turn implies limited to no borrowing in equilibrium.

4.2 Heterogeneous beliefs, natural borrowing limits

Sandroni (2000) shows that agents with more accurate forecasts survive in the limit. Those with less accurate beliefs will be driven out of the market. Blume & Easley (2006) show that this result extends to all the economies in which the allocation is Pareto optimal and to the environments with learning. Unfortunately, in the limit only the agent with the more accurate beliefs survives.[4] So, even though these economies have the potential to deliver interesting asset market predictions they are non-stationary. Blume & Easley (2009) write:

> Analysis of [infinite horizon, stochastic, general equilibrium economies with heterogeneous agents] are beginning to appear in both finance and macroeconomics in response to the inability of representative agent models to fit asset pricing and macro data. ... [But] we do not expect them to stand up to market selection.

A stationary equilibrium may be guaranteed to exist by limiting agents' borrowing to prevent them from loosing all their wealth. Imposing an arbitrary exogenous borrowing limit is not attractive as solution depends crucially on the imposed limit. This issue is solved by adding an endogenous borrowing/participation constraint.

4.3 Two-period analytic example

We now present a two-period model to build intuition about how heterogeneous beliefs affect the equilibrium outcome. We let the two individuals to be completely symmetric in period 0. Both agents receive one unit of good and start with zero assets:

$$(e^1(s_0), e^2(s_0)) = (1, 1), \quad (a^1(s_0), a^2(s_0)) = (0, 0).$$

Individual's endowments are "symmetric"; so, instead of a competitive equilibrium problem we can solve for a Pareto problem with equal weights on the two agents. We assume that $u(c) = (c^{1-\gamma} - 1)/(1 - \gamma)$. Let $\bar{e}(s) = 0.5(e^1(s) + e^2(s))$ denote the per capita endowment in state s.

First, let's analyze the optimal allocation. Agent a consumes 1 unit of good in period 1 and

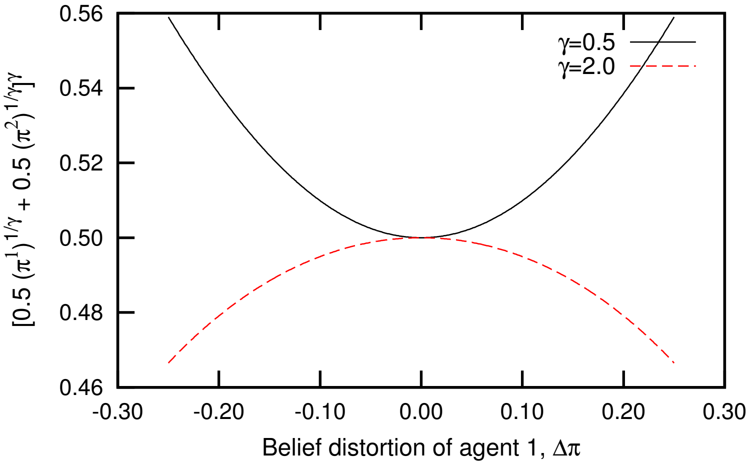
$$c_1^a(s) = \bar{e}_1(s) \frac{(\pi^a(s))^{1/\gamma}}{0.5(\pi^1(s))^{1/\gamma} + 0.5(\pi^2(s))^{1/\gamma}}$$

in period 1, state s. Agent 1's purchase of security s is

$$a^1(s) = (\bar{e}_1(s) - e_1^1(s)) + \bar{e}_1(s) \frac{(\pi^1(s))^{1/\gamma} - (\pi^2(s))^{1/\gamma}}{(\pi^1(s))^{1/\gamma} + (\pi^2(s))^{1/\gamma}}$$

[4] If there are several agents with equally accurate beliefs then the log ratio of marginal utilities is a mean-zero random walk.

Figure 4: Belief aggregator for "symmetric" beliefs



The first term is standard and reflects desire to hedge income risk. The second term

$$x^1(s) \equiv \bar{e}_1(s)\frac{(\pi^1(s))^{1/\gamma} - (\pi^2(s))^{1/\gamma}}{(\pi^1(s))^{1/\gamma} + (\pi^2(s))^{1/\gamma}},$$

is the speculative trade that is proportional to the difference in beliefs adjusted for risk tolerance. When do the hedge and the speculative trade have the same sign? When an agent is pessimistic. That is he must put a lower probability than the other agent on states when his or her income is above the population average.

Second, let's analyze the equilibrium price system. Price of an Arrow security paying is state s is

$$Q(s) = \beta(\bar{e}_1(s))^{-\gamma}[0.5(\pi^1(s))^{1/\gamma} + 0.5(\pi^2(s))^{1/\gamma}]^\gamma,$$

where instead of the true probability $\pi(s'|s)$ we have a CES aggregator of population beliefs. Heterogeneity of beliefs across idiosyncratic states has limited effect in this two period example. When beliefs are correct on average (across population), $(\pi^1(s), \pi^2(s)) = (\pi(s) + \Delta\pi, \pi(s) - \Delta\pi)$ where $\Delta\pi$ lies in a small open interval centered at 0, the effect on asset prices is $O(|\Delta\pi|^3)$ (see figure 4). Finally, note that with $\gamma > 1$, as we assume later, belief heterogeneity depresses equilibrium prices.

5 Recursive Formulation

Consider a decentralized environment in which agents face endogenous borrowing constraints. Let $\omega = (\omega^1, ..., \omega^I)$ be a distribution of financial wealth in the economy. The state vector is (ω, s). Agents rationally expect that wealth distribution in state s' tomorrow will be $\Omega(\omega, s, s')$.

To formulate the problem of an agent recursively we need to normalize variables by the current level of the aggregate income $y(s^t)$. Following remark 1 above we normalize value functions by $y(s^t)^{1-\gamma}$.

Let $Q(\omega, s, s')$ be the price of a security paying one unit of good in state s' tomorrow when the wealth distribution is ω and the current state is s. Let $V^i(a, \omega, s)$ be the optimal *normalized* life-time utility of an agent i who has assets a and when the economy's state is (ω, s). The value function must satisfy the following Bellman equation:

$$V^i(a, \omega, s) = \max_{c, a'(s')} \Big[u(c) + \beta \sum_{s'} V^i(a'(s'), \Omega(\omega, s, s'), s') \lambda(s')^{1-\gamma} \pi^i(s'|s) \Big] \tag{6}$$

subject to the *normalized* budget constraint

$$c + \sum_{s'} Q(\omega, s, s') \lambda(s') a'(s') = \eta^i(s) + a, \quad \forall t \tag{7}$$

and a borrowing limit

$$a'(s') \geqslant -B^i(\Omega(\omega, s, s'), s'), \quad \forall s' \in \mathcal{S}. \tag{8}$$

The endogenous borrowing limit $B^i(\omega, s, s')$ is implicitly defined by the following relation:

$$V^i(-B^i, \Omega(\omega, s, s'), s') = V^i_{aut}(s'), \quad \forall s'. \tag{9}$$

It is easy to show that V^i is increasing in a.[5] So, the borrowing constraint is equivalent to:

$$V^i(a'(s'), \Omega(\omega, s, s'), s') \geqslant V^i_{aut,T}(s'), \quad \forall s', \tag{10}$$

where

$$V^i_{aut,T}(s) = E\Big[\sum_{t=0}^{T-1} \beta^t u(y_t) + \beta^T V^i(0, \omega_T, s_{T-1}, s_T) | s \Big], \quad \forall s, \tag{11}$$

[5] This follows from the monotonicity of the budget set.

We will often refer to this constraint as *participation constraint*. In this formulation, an individual is banned from the market only for T periods. When $T = \infty$ then an individual never returns to the market.

A *recursive competitive equilibrium* (RCE) is a list of functions $Q(\omega, s, s')$, $\{B^i(\omega, s)\}_{i\in\mathcal{I}}$, $\{c^i(a, \omega, s), a^{i\prime}(a, \omega, s, s')\}_{i\in\mathcal{I}}$ such that:

a) given the price system Q and borrowing limit B^i policy functions $(c^i, a^{i\prime})$ solve agent i's optimization problem;

b) good market clears:

$$\sum_{i=1}^{I} c^i(a^i, \omega, s) = \sum_{i=1}^{I} y^i(s); \tag{12}$$

c) financial markets clear:

$$\sum_{i=1}^{I} a^{i\prime}(a^i, \omega, s, s') = 0, \quad \forall s' \in \mathcal{S}; \tag{13}$$

d) wealth evolution is consistent with individual decisions:

$$\omega = (a^1, ..., a^I) \tag{14}$$

$$\Omega(\omega, s, s') = (a^{1\prime}(a^1, \omega, s, s'), ..., a^{I\prime}(a^I, \omega, s, s')), \quad \forall s' \in \mathcal{S}. \tag{15}$$

5.1 Binding borrowing limits

With homogenous information borrowing limits may remain slack in equilibrium. For example, when the agents are infinitely patient ($\beta = 1$) or the agents' income is very volatile ($std(y^i(s)) \gg 0$) they will never choose to default. The reason for this is a declining attractiveness of the financial autarky. In the environment with heterogeneous beliefs borrowing limits always bind. This is a consequence of the survival result in Blume and Easley (2006). If the borrowing limit did not bind consumption of an individual with the lowest survival index must converge to zero. This means that at least in the limit the value of staying in the financial market is lower than the value of financial autarky. The latter is not possible with limited participation. We formalize this result with a proposition.

Proposition 1. *Suppose $u(0) = -\infty$. If there exist $i, j \in \{1, .., I\}$ such that $\Pi^i \neq \Pi^j$ then borrowing limits are binding in equilibrium for either agent i or agent j.*

Proof. First, because endowments are bounded away from zero the autarky value is finite for each agent and state.

Next, without loss of generality, suppose that agent 1 and 2 have different beliefs. If agent 1's beliefs are less accurate as measured by the relative entropy then $\limsup_{t\to\infty} c_t^1 = 0$ by theorem 2 of Blume & Easley (2006). Because u is continuous, this implies that $\limsup_{t\to\infty} U_t^1 = u(0)/(1-\beta) < V_{aut}^1(1)$, where U_t^i denotes the continuation utility of agent i starting from date t.

If agent 1 and agent 2 hold equally accurate beliefs then $\liminf_{t\to\infty} c_t^1 = 0$ (see Blume & Easley (2009), p.11). This implies that for any $\varepsilon > 0, c_t^1 < \varepsilon$ infinitely often. Choose ε such that $\bar{U}(\varepsilon) \equiv u(\varepsilon) + \beta u(\max_s[e(s)])/(1-\beta) = 0.5 \min_s[V_{aut}^1(s)]$. Then $U_t^1 < \bar{U}(\varepsilon) < V_{aut}^1(s), \forall s$ infinitely often. So, the participation constraint must be binding as $V_{aut}^1(1) > -\infty$. □

5.2 Solution

In general it is not possible to solve for the RCE analytically. We compute the model solution numerically for the case with two agents, $I = 2$. In this case the state vector contains only one continuous state variable: financial wealth of the first agent, ω^1. Financial wealth of the second agent can be obtained via the market clearing condition, $\omega^2 = -\omega^1$. This simplification makes the problem very tractable. Because in equilibrium $\omega \equiv a^1 = -a^2$ we can state the system of equilibrium conditions in terms of auxiliary value functions:

$$
\begin{aligned}
v^1(\omega, s) &= V^1(\omega, \omega, s), \\
v^2(\omega, s) &= V^2(-\omega, \omega, s).
\end{aligned}
$$

Then the solution to the model is fully characterized by the following system of equations: $\forall \omega, s, s'$,

$$Q(\omega, s, s') = \beta \pi^i(s'|s) g(s') \frac{u'(c^i(\Omega(\omega, s, s'), s')) + \mu^i(\omega, s, s')}{u'(c^i(\omega, s))}, \quad i \in \mathcal{I}, \tag{16a}$$

$$y^1(s) + \omega = c^1(\omega, s) + \sum_{s'} Q(\omega, s, s') g(s') a^{1\prime}(\omega, s, s'), \tag{16b}$$

$$y^2(s) - \omega = c^2(\omega, s) + \sum_{s'} Q(\omega, s, s') g(s') a^{2\prime}(\omega, s, s'), \tag{16c}$$

$$\Omega(\omega, s, s') = a^{1\prime}(\omega, s, s'), \tag{16d}$$

and

$$v^i(\omega, s) = u(c^i(\omega, s)) + \beta \sum_{s'} v^i(\omega', s') g(s') \pi^i(s'|s), \quad i \in \mathcal{I}, \tag{16e}$$

$$v^i(\omega', s') \geqslant V^i_{aut}(s'), 0 = \mu^i(\omega, s, s')(v^i(\omega', s') - V^i_{aut}(s')), \quad i \in \mathcal{I}. \tag{16f}$$

The first four equations are enough to solve for an equilibrium with natural borrowing constraints.[6] The last two equations allow us to keep track of the endogenous borrowing limits. We compute the RCE using the following algorithm.

1. Fix borrowing limits.
2. Solve iteratively the system of first-order conditions (16).
3. Compute value functions and update borrowing limits.[7]
4. Repeat the procedure until borrowing limits on step 1 and 3 are sufficiently close.

On step 2 we solve the following system of equations:

$$\Phi(c, a', \omega, s | \rho^c, \rho^a) = 0$$
$$\Omega(\omega, s, s') = a^{1\prime}(s')$$

where ρ_c, ρ_a are current policy functions. The solution to this system consists of two functions:

$$c = \hat{\rho}^c(\omega, s), \quad a'(s') = \hat{\rho}^a(\omega, s, s').$$

We use this solution to update the initial policy functions. The solution is projected on the space of cubic splines and we iterate until $\sup_{\omega,s} |\hat{\rho}(\omega, s) - \rho(\omega, s)| < 10^{-6}$.

[6]Natural borrowing constraints are defined by: $B^i(s^t) = \sum_\tau \sum_{s^\tau} p^t_\tau(s^\tau) e^i(s^\tau)$.

[7]Update rule that we use is $B^{n+1} = \alpha B^n + (1-\alpha) B'$, where B' is the new borrowing limit and B^n is the borrowing limit used on iteration n. We set $\alpha = 0.5$.

5.3 Asset prices

Since we have a full set of Arrow securities traded in equilibrium we can price any asset. Let $p^d(\omega, s)$ be the ex-dividend price of an asset paying the dividend $d(s)$ when the economy's state is (ω, s). This price can be represented recursively as follows:

$$p^d(\omega, s) = \sum_{s'} Q(\omega, s, s')(p^d(\Omega(\omega, s, s'), s') + d(s')). \tag{17}$$

Realized return on the asset is:

$$R(\omega, s, s') = \frac{p^d(\Omega(\omega, s, s'), s') + d(s')}{p^d(\omega, s)}. \tag{18}$$

"Stock market index" is the asset paying the aggregate endowment. Long-term bond pays one unit of good in each period and state. Short-term (one period) bond pays one unit of good in each state tomorrow. We denote the prices of these assets by $p^s(\omega, s), p^{bl}(\omega, s)$ and $p^{bs}(\omega, s)$ respectively.

When borrowing constraints do not bind. It is possible to characterize Arrow security prices analytically when borrowing limits are not binding. Thus an Arrow security paying one unit of good in state s' tomorrow conditional on the today's aggregate state (ω, s) is:

$$Q(\omega, s, s') = \beta g(s')^{-\gamma} \tilde{\pi}(s'), \tag{19}$$

where

$$\tilde{\pi}(s') = \Big[\sum_i c^i(\omega, s) \pi^i(s'|s)^{1/\gamma}\Big]^{\gamma}. \tag{20}$$

The term $\beta g(s')^{-\gamma}$ is standard. The term $\tilde{\pi}(s')$ is the CES aggregate of individuals' beliefs in the economy as in section 3.3. It is not a measure unless $\gamma = 1$. Fluctuations in the wealth distribution trigger movements in consumption shares. The latter in turn affects how individual beliefs are aggregated into prices.

5.4 Two-state example

We use a two-state example to demonstrate the model's capability to solve asset pricing puzzles.[8] Suppose s_t is a symmetric two-state first-order pro-

[8]This is also an excellent test case: for all th experiments below the two moments of the numerically computed pricing kernel coincide with the analytically computed moments.

cess with $prob(s_i|s_i) = 0.75, i = 1, 2$. Endowments are given by:

$$(e^1(z), e^2(s)) = \begin{cases} (0.64, 0.36), & \text{if } s = 1 \\ (0.36, 0.64), & \text{if } s = 2 \end{cases}. \tag{21}$$

Log of the individual income share has mean 0.50, standard deviation 0.30 and autocorrelation 0.50. This is also the example studied in Alvarez & Jermann (2001). Our task is to analyze the pricing kernel. In particular, we are interested if it satisfies the Hansen-Jaganathan (1991, henceforce, HJ) bound. This bound is a bare minimum that any pricing kernel should satisfy. Figure 5 plots HJ bounds.

Point A_0 refers to the point considered by Alvarez & Jermann (2001). They demonstrate that when $\beta = 0.65$ and $\gamma = 2.00$ then the pricing kernel from the model with the endogenous solvency constraints satisfies these bounds. The low discount factor was chosen to make the autarkic allocation a very good alternative that restricts optimal risk-sharing. So, how restrictive is the solvency constraint? It turns out that for this parametrization borrowing is essentially assumed away. Maximum borrowing is 0.0948% of an average income.

Consider further a scenario with $\beta = 0.70$ when the financial autarky is a less attractive alternative. In this case the pricing kernel does not satisfy the HJ bound (see point A_1); this statement is true irrespectively of γ. And for $\beta = 0.80, \gamma = 2.00$ full risk-sharing is attainable and, so, the pricing kernel degenerates to a constant β (point A_2).

Let us now consider a simple form of belief heterogeneity. In the specification below an agent is pessimistic when his or her income is low:

$$P^1 = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 - dp & 0.75 + dp \end{bmatrix}, \quad P^2 = \begin{bmatrix} 0.75 + dp & 0.25 - dp \\ 0.75 & 0.25 \end{bmatrix}. \tag{22}$$

The special case with homogeneous beliefs obtains for $dp = 0$. When $dp < 0$ individuals are optimistic. With heterogeneous beliefs full risk sharing is not attainable for any $\gamma > 0$. Yet, even for small deviation from homogeneous beliefs, $dp = 0.01$ allows to bring the model into compliance with the HJ bounds. Points B_1 and B_2 are counterparts of points A_1 and A_2 respectively. Note also that equilibrium allocation in the models B_1 and B_2 is distinct from the autarkic one. The maximum amount borrowed is above 10% of current income. These findings are summarized in table 1.

Figure 5: Hansen-Jaganathan bounds

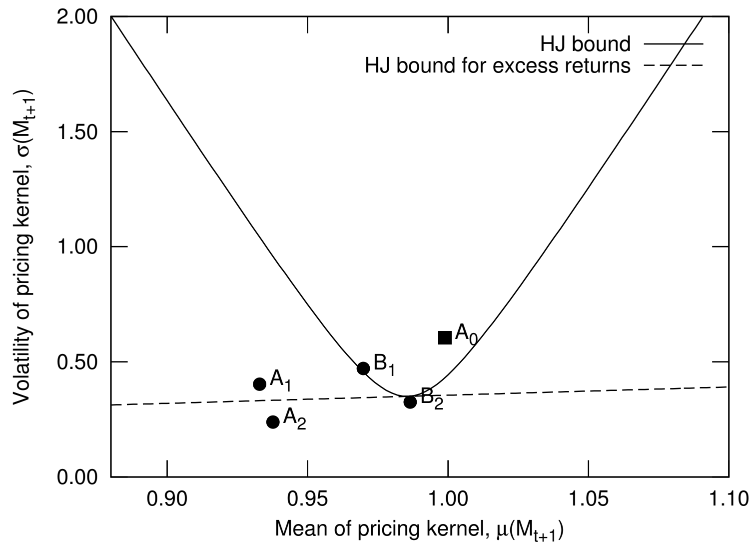


Table 1: Equilibria in two-state model

	β	γ	dp	borrowing
A_0	0.65	2.00	0.00	0.0005
A_1	0.70	2.00	0.00	0.0405
B_1	0.70	2.00	0.01	0.0362
A_2	0.80	1.50	0.00	0.0725
B_2	0.80	1.50	0.01	0.0484

6 Calibration and numerical results

We need to calibrate two sets of parameters: process for aggregate growth and process for individual income shares, preferences. We describe each next.

Process for aggregate income growth. We use Mehra & Prescott (1985) specification but drop the assumption that the distribution is symmetric. In particular, following Alvarez & Jermann (2001) we add a restriction that under the ergodic distribution probability of a boom is 2.65 higher than that of a recession. The resulting process is a two-state Markov process with states $\{0.9961, 1.0761\}$ and transition matrix Π_g:

$$\Pi_g = \begin{bmatrix} 0.6877 & 0.3123 \\ 0.8277 & 0.1723 \end{bmatrix}. \tag{23}$$

Ergodic probability of the boom state is 0.7260.

Process for individual income shares. Process for individual income shares is assumed to be independent of the aggregate growth process. Heaton & Lucas (1996) estimate it to be an AR(1) with persistence 0.530 and standard deviation 0.296. The resulting process is a two-state Markov process with states $\{0.3562, 0.6438\}$ and transition matrix Π_η^0:

$$\Pi_\eta^0 = \begin{bmatrix} 0.765 & 0.235 \\ 0.235 & 0.765 \end{bmatrix}. \tag{24}$$

Individuals differ in their perception of Π_η. We assume that individual i believes that the transition matrix is:

$$\Pi_\eta^i = \begin{bmatrix} 0.765 - dp^i & 0.235 + dp^i \\ 0.235 - dp^i & 0.765 + dp^i \end{bmatrix}. \tag{25}$$

Individual i is optimistic if $dp^i > 0$ and pessimistic otherwise. Disagreement is only about the evolution of individuals' own income share. *Everyone agrees on the evolution of the aggregate growth rate.*

The exogenous aggregate state of the economy s is a first-order Markov process with four states:

$$(g(s), \eta^1(s), \eta^2(s)) = \begin{cases} (0.9961, 0.3562, 0.6438), & s = 1 \\ (0.9961, 0.6438, 0.3562), & s = 2 \\ (1.0761, 0.3562, 0.6438), & s = 3 \\ (1.0761, 0.6438, 0.3562), & s = 4 \end{cases}, \tag{26}$$

and a transition matrix $\Pi_g \otimes \Pi_\eta^0$. Moments that were used to calibrate this process are summarized in table 2. Moments M1-M6 are the same as in Alvarez & Jermann (2001).[9]

Preference parameters and bounds on belief heterogeneity. We have three preference parameters: $\beta, \gamma, dp^1 = -dp^2$. These parameters are set to match

[9] We do not use their moments M7-M10. So, our parametrization corresponds to their 'homoscedasticity' case.

Table 2: Moments used for calibration of income processes

Moment	Value
M1. Growth rate expectation	1.83
M2. Growth rate st.deviation	3.57
M3. Growth rate persistence	-0.14
M4. Likelihood of boom vs recession	2.65
M5. Income share st.deviation	0.296
M6. Income share persistence	0.530

the asset pricing data: average risk-free rate, average risk premium and the respective standard deviations. We equally weight each data moment. We restrict the parameter space to $[0.5, 1.0] \times [1.0, 5.0] \times [0.0, \overline{dp}]$.

We set $\overline{dp}$ in the following way. Suppose that an individual has a time series of length 100. We ask what is the probability that the data will allow him or her to detect that beliefs are wrong? We imagine that an individual uses likelihood ratio to test two hypotheses: $H_0 : (\pi_{11}, \pi_{22}) = (\pi^i_{11}, \pi^i_{22})$ vs $H_1 : (\pi_{11}, \pi_{22}) = ((\pi^0_{11}, \pi^0_{22})$. The individual rejects H_0 (his beliefs) in favor of the empirical estimate $\hat{\rho}$ when the likelihood ratio is below a cutoff value. We choose a cutoff value that is 10% percentile of the likelihood ratio statistic. The cutoff is chosen so that the individual would reject his dogmatic beliefs with probability 90%. The set of beliefs that would be rejected by the hypothetical econometrician is plotted in figure 6.

We set $\beta = 1.0, \gamma = 1.98$ and $dp = 0.025$. Under these parameters the model with heterogeneous beliefs and endogenous borrowing constraints performs the best. Different specifications that we contemplate are described in table 3. Simulation results are reported in table 4. Our benchmark specification P1 does fairly well in matching the asset price data. It should be compared to specification P2 where we switch off belief heterogeneity. Even though we obtain equity premium which is only 2.87% we match equity return volatility nearly perfectly. We also get much closer to the observed statistics on trading volume. The specification with complete markets (AD) does horribly matching asset price data as was observed by previous researchers. Yet, it beats model P2 when it comes to trading volume.

Wealth dynamics. Figure 7 plots time path of the agent 1's wealth under different scenarios for a random but fixed sequence of shocks. Under complete markets when the two agents hold correct beliefs wealth of agent

Figure 6: Set of beliefs that can be identified to be incorrect

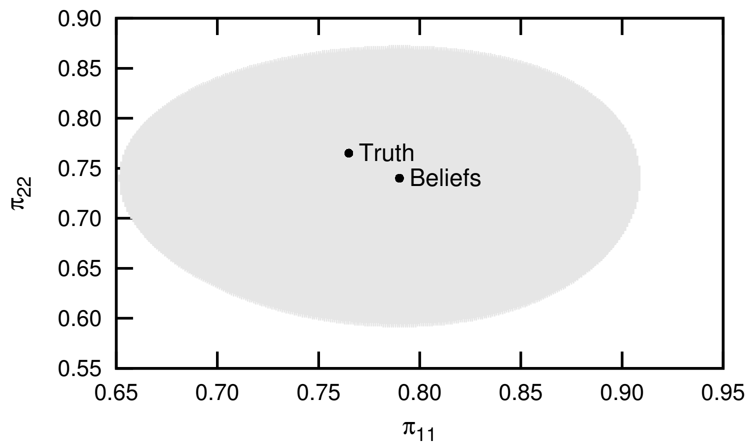


Table 3: Competing model specifications

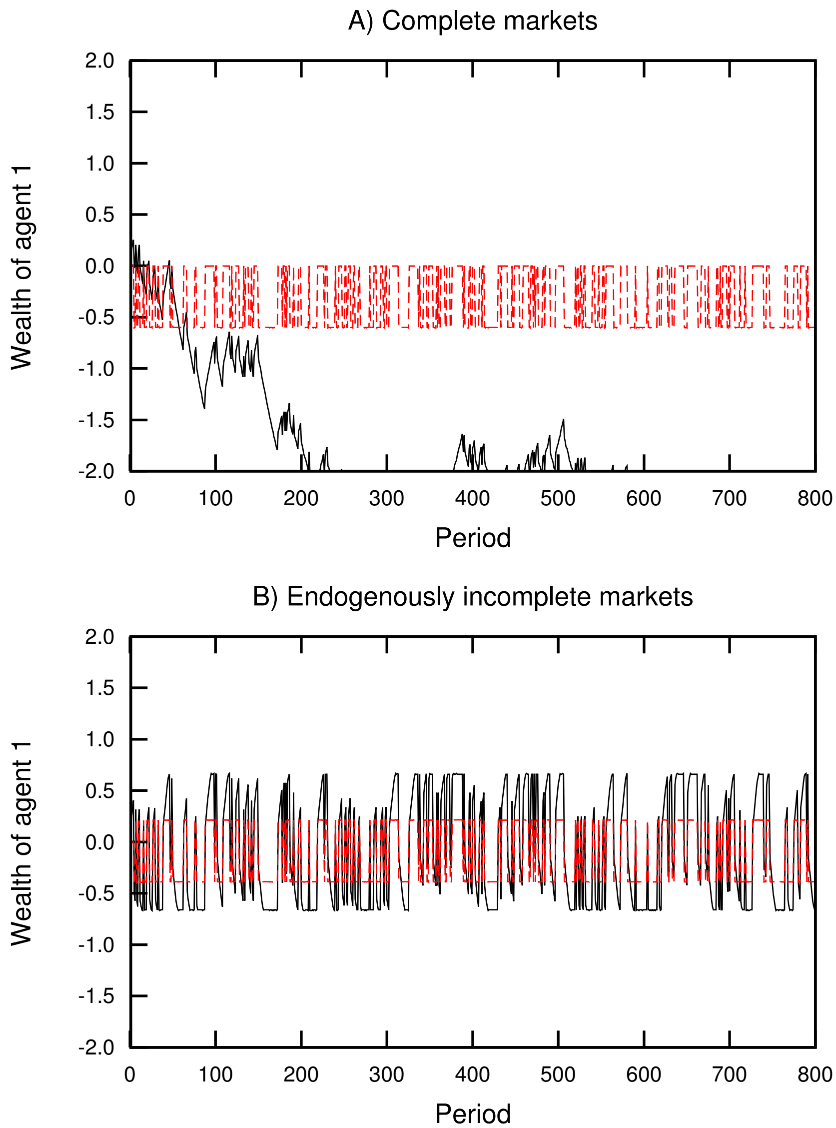
Spec	Preferences	T	dp
P1	$\beta = 1.00, \gamma = 1.98$	T=10	0.025
P2	$\beta = 1.00, \gamma = 1.98$	T=10	–
P3	$\beta = 1.00, \gamma = 1.98$	T=∞	0.025
AD	$\beta = 1.00, \gamma = 1.98$	–	–

Table 4: Model moments

Statistic	Data	P1	P2	AD	P3
$E(r^f)$	0.80	0.80	1.25	3.30	2.01
$E(r^p)$	6.18	2.87	3.84	0.25	1.42
$sd(r^f)$	5.67	3.42	3.72	0.41	2.43
$sd(r^e)$	16.54	16.11	20.20	0.94	11.94
arch(r^e)	0.25	0.28	0.15	0.00	0.46
$sd(v)$	15.48	17.16	8.63	7.65	0.43
$cor(v, y)$	-0.01	0.01	0.50	0.03	-0.03
$cor(v, r^e)$	0.08	0.09	0.22	0.00	0.03
Tightest BL/income	<2.0	1.03	0.62	24.58	1.86

1 oscillates in a narrow range (panel a/b, dashed line). With heterogeneous

Figure 7: Wealth dynamics under complete and incomplete markets



beliefs wealth of the agents oscillates more widely (panel a, solid line). That is a consequence of the survival analysis in Blume & Easley (2006): wealth of any individual drops to arbitrarily small number infinitely often. Under endogenously incomplete markets with heterogeneous beliefs wealth fluctuates in wider range. Also due to belief heterogeneity wealth can now take a continuum of values instead of just 4 (number of exogenous states).

So, three facts emerge from the figure. First, under homogenous beliefs wealth is very stable. (In fact, wealth distribution is strongly stationary – that is it is only a function of the exogenous state.) Second, under complete markets but heterogeneous beliefs wealth is non-stationary. The solid line series depicting wealth path in panel (a) walks slowly between $[-29.37, 29.37]$. Third, under endogenously incomplete markets and heterogeneous beliefs wealth is volatile but stationary. With incomplete markets agents are never allowed to accumulate "too much" debt because they can repudiate their obligations. This limits the positions that market participants can take. In the incomplete markets model the most that the agent can borrow/owe is approximately 0.66 in low income state and 0.75 in high income state. The natural borrowing limit in this environment is approximately 29.40,[10]

Asset returns. Figure 8 plots asset returns. With complete markets the asset returns with or without heterogeneous beliefs are nearly identical (only the first is plotted). However, with endogenous borrowing limits assets returns are 16.33 times more volatile. Note also a characteristic volatility clustering of stock returns. Volatility clustering occurs because asset price volatility increases with the dispersion of wealth. Endogenously incomplete markets amplify volatility clustering nearly two-fold as measure by the ARCH coefficient for the estimated equity returns process.

Interestingly, that heterogeneous beliefs *per se* do not have a *direct* effect on asset prices. It only happens through the wealth distribution. Note that both a risk-free bond and an equity state pay only conditional on the aggregate growth rate. That is if beliefs were homogeneous individual income distribution would not affect the asset prices. With heterogeneous beliefs asset price dynamics are more complicated. For example, when economy switches from state 1 to state 2 the aggregate growth rate remains the same. Yet, individuals' income distribution changes. Since individuals are pessimistic about their incomes individual 1 invests more in security paying in state 1 and individual 2 in state 2. If individuals start in $(\omega, s) = (0, 1)$ and

[10] We say "approximately" because there is a different limit for each state.

Figure 8: Asset returns under complete and incomplete markets

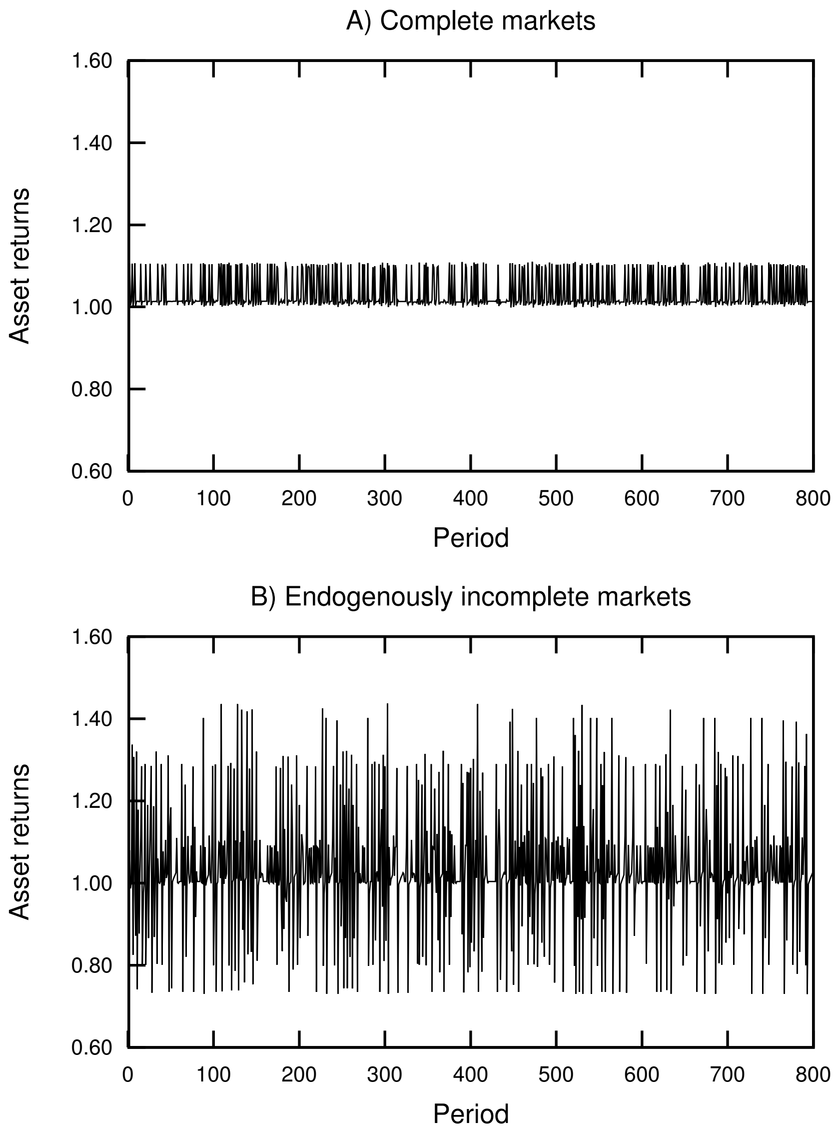
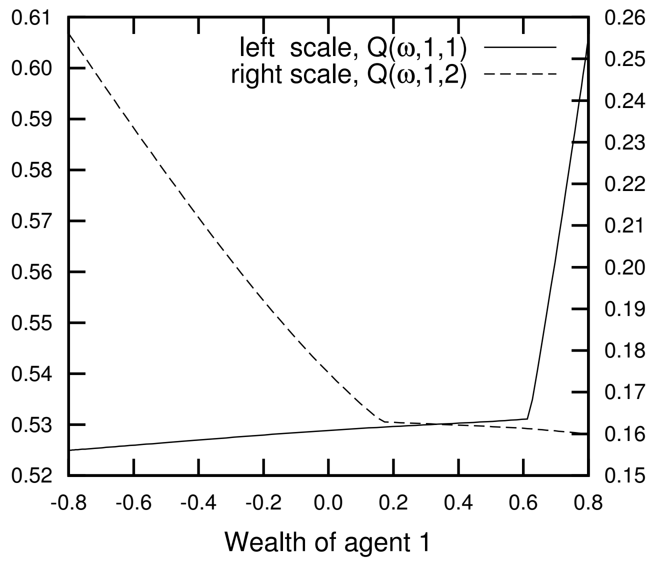


Figure 9: Arrow security prices (P1)



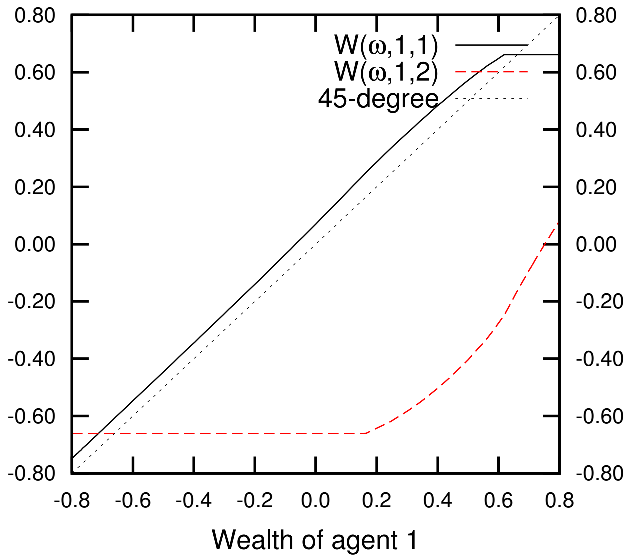
state 1 realizes tomorrow individual 1 gains wealth and equilibrium prices put more weight on beliefs of individual 1. For this reason price $Q(\omega, 1, 1)$ shown in figure 9) is increasing and price $Q(\omega, 1, 2)$ is decreasing. This force is equilibrating: an individual gaining wealth has to pay more for assets that pay in states he thinks are more likely.

Now suppose that state 1 occurs repeatedly for several periods. That is economy grows at high speed but individual 1 receives low income share. In this case individual 1's wealth grows (see figure 7). Eventually, individual 2 will become borrowing constrained and will want to default. For the individual 2 to stay in the market prices need to go up. That is return in this state goes down making it easier for agent 2 to roll over debt and less attractive for individual 1 to invest in state 1.

The role of heterogeneous beliefs is to amplify wealth dynamics. The latter, in turn, never disappears because the *endogenous* borrowing constraints keep the agent trading in the market. When agent 1 looses wealth prices on the Arrow securities change favorably and so he or she can increase wealth. Similar forces exist even with natural borrowing limit but they are too large to let the agent accumulate new wealth quickly.

Trading volume. As in the data we measure trading volume as the ab-

Figure 10: Wealth transition (P1)



solute value of all transactions in any period:

$$vol_t = \sum_j Q_j |a_j^1|.$$

With homogenous beliefs trading volume is constant as long as the state does not change. When state changes individuals change their portfolio. Thus trading volume is as persistent as is the exogenous state. For the same reason trading volume is highly correlated with aggregate income (and consumption) which is counterfactual. Finally trading volume is bounded above by the income differences between individuals.

With heterogeneous beliefs individuals trade also for speculative reasons. Trading volume is largest when wealth distribution reaches one of the extremes. Trading volume is uncorrelated with the rest of the economy because it is largely driven by luck.

7 Relation to model uncertainty

The parametrization that we chose presents us with an opportunity to highlight the relation between heterogeneous beliefs and model uncertainty. Consider setting (1) when agents are pessimistic about their own income processes and setting (2) when agents face model uncertainty about the individual income shares but not the aggregate income process. We could say that agents in setting 92) have multiplier preferences as in Hansen and Sargent. Because we there are only two possible income distributions the two settings are fully equivalent. That is we could interpret our environment as the one in which agents have correct beliefs but face model uncertainty and therefore endogenously behave as if they where pessimistic.

8 Concluding Remarks

We build a simple model of with diverse beliefs and incomplete markets. Markets are incomplete because agents face borrowing limits that could be endogenous or exogenous. We study simple exogenous borrowing limits and endogenous solvency constraints. In addition to matching standard asset pricing facts, the model generates volatility clustering in asset returns and trading volume. Trading volume is persistent and volatile, and uncorrelated with asset returns and the macroeconomic fundamentals as observed in the data.

Borrowing limits is probably the simplest way to induce market incompleteness. And it seems a natural step to take. But it is important to explore model predictions in the setup with a more realistic asset structure, say a stock and a bond. Such realism could enable to study endogenous market closures among other things. Finally, it is interesting to explore the relation between the diverse beliefs and recursive utility in the context of incomplete financial markets.

References

Alvarez, F. & Jermann, U. (2000), 'Efficiency, equilibrium, and asset pricing with risk of default', *Econometrica* **68**, 775–797.

Alvarez, F. & Jermann, U. (2001), 'Quantitative asset pricing implications of endogenous solvency constraints', *Review of Financial Sudies* **14**, 1117–1151.

Blume, L. E. & Easley, D. (2006), 'If you're so smart, why aren't you rich? belief selection in complete and incomplete markets', *Econometrica* **74**(4), 929–966.

Blume, L. E. & Easley, D. (2009), 'Heterogeneity, selection and wealth dynamics', *Cornell University manuscript* .

Blume, L. E., Easley, D. & O'Hara, M. (1994), 'Market statistics and technical analysis: The role of volume', *Journal of Finance* **49**, 153–181.

Cogley, T. & Sargent, T. J. (2009), 'Diverse beliefs, survival, and the market price of risk', *Economic Journal* **119**, 354–376.

Cogley, T., Sargent, T. & Tsyrennikov, V. (forthcoming), 'Wealth dynamics in a bond economy with heterogeneous beliefs', *Economic Journal* .

Friedman, M. (1953), 'Choice, chance, and the personal distribution of income', *Journal of Political Economy* **61**(4), 277–290.

Harrison, M. & Kreps, D. (1978), 'Speculative investor behavior in a stock market with heterogeneous expectations', *Quarterly Journal of Economics* **92**(2), 323–336.

He, H. & Wang, J. (1995), 'Differential information and dynamic behavior of stock trading volume', *Review of Financial Studies* **8**(4), 919–972.

Heaton, J. & Lucas, D. (1996), 'Evaluating the effects of incomplete markets on risk sharing and asset pricing', *Journal of Political Economy* **104**(3), 443–487.

Judd, K., Kubler, F. & Schmedders, K. (2003), 'Asset trading volume with dynamically complete markets and heterogeneous agents', *Journal of Fiannce* **58**, 2203–2218.

Kehoe, T. J. & Levine, D. K. (1993), 'Debt-constrained asset markets', *Review of Economic Studies* **60**, 865–888.

Kocherlakota, N. (1996), 'Implications of efficient risk sharing without commitment', *Review of Economic Studies* pp. 595–609.

Lucas, R. E. (1978), 'Asset prices in an exchange economy', *Econometrica* **46**, 1429–1445.

Mehra, R. & Prescott, E. C. (1985), 'The equity premium: A puzzle', *Journal of Monetary Economics* **15**(2), 145–162.

Sandroni, A. (2000), 'Do markets favor agents able to make accurate predictions?', *Econometrica* **68**(6), 1303–1341.

Veronesi, P. (2000), 'How does information quality affect stock returns?', *Journal of Finance* **55**(2), 807–837.

A Data sources

Industrial production is obtained from Board of Governors: series code is B50001. Real variables are computed by deflating nominal variables with the CPI. The latter is obtained from FRED: series code is CPIAUCNS, "Consumer Price Index for All Urban Consumers: All Items, Not Seasonally Adjusted".

Entries in Compustat database are described in Table 5. Entry `permno` is a unique company identifier. A measure of trading activity is obtained by multiplying the number of shares traded, `vol`, and the transaction price, `vol`: $\texttt{trd}_t = \texttt{vol}_t \cdot \texttt{prc}_t$. The data in Compustat is daily. So, we aggregate the data into monthly series simply by summing the value of all transactions in a given month.

Entry	Desciption
date	Date in 'yyyymmdd' format
ticker	Company's ticker
permno	Company's permanent number
shrout	Outstanding number of shares
prc	Share price[a]
vol	Trading volume

[a] if positive it is the last transaction price; otherwise it is an average of close bid and ask prices.

Table 5: Compustat database